Introduction to Bubble Sort and its Variants

The sequential bubble sort algorithm compares and exchanges adjacent elements in the sequence to be sorted:

```
1.    procedure BUBBLE_SORT(n)
2.    begin
3.        for i := n - 1 downto 1 do
4.            for j := 1 to i do
5.                compare-exchange(a_j, a_{j+1});
6.    end BUBBLE_SORT
```

Sequential bubble sort algorithm.

Bubble Sort and its Variants

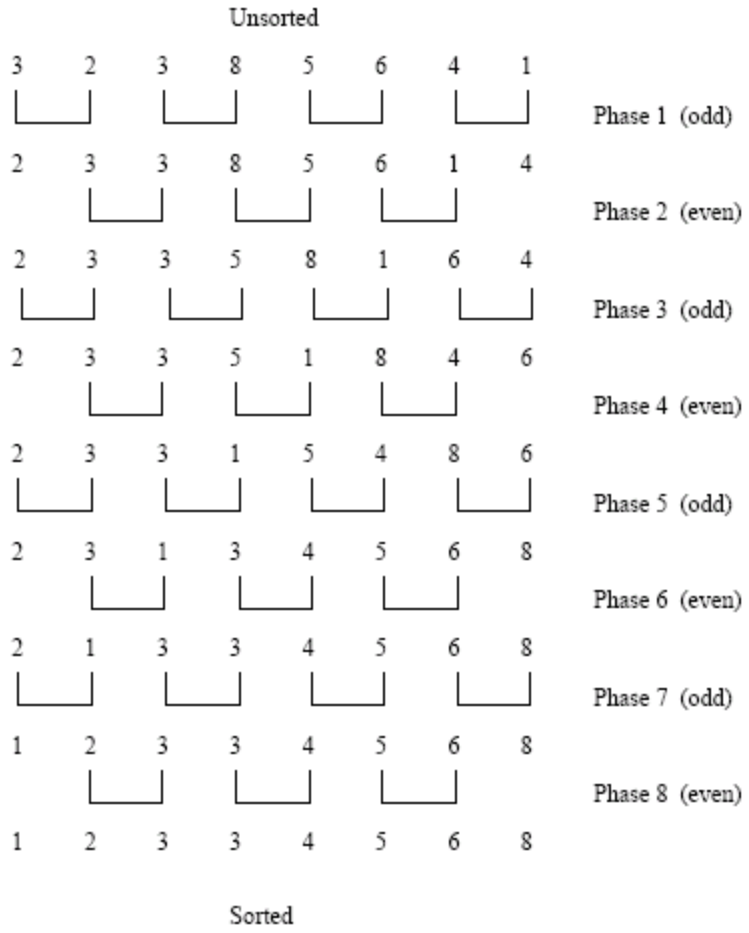
- The complexity of bubble sort is $\Theta(n^2)$.
- Bubble sort is difficult to parallelize since the algorithm has no concurrency.
- A simple variant, though, uncovers the concurrency.

Odd-Even Transposition

```
1.   procedure ODD-EVEN(n)
2.   begin
3.       for i := 1 to n do
4.       begin
5.           if i is odd then
6.               for j := 0 to n/2 − 1 do
7.                   compare-exchange(a_{2j+1}, a_{2j+2});
8.           if i is even then
9.               for j := 1 to n/2 − 1 do
10.                  compare-exchange(a_{2j}, a_{2j+1});
11.      end for
12.  end ODD-EVEN
```

Sequential odd-even transposition sort algorithm.

Odd-Even Transposition



Sorting $n = 8$ elements, using the odd-even transposition sort algorithm. During each phase, $n = 8$ elements are compared.

Odd-Even Transposition

- After n phases of odd-even exchanges, the sequence is sorted.
- Each phase of the algorithm (either odd or even) requires $\Theta(n)$ comparisons.
- Serial complexity is $\Theta(n^2)$.

Parallel Odd-Even Transposition

- Consider the one item per processor case.
- There are n iterations, in each iteration, each processor does one compare-exchange.
- The parallel run time of this formulation is $\Theta(n)$.
- This is cost optimal with respect to the base serial algorithm but not the optimal one.

Parallel Odd-Even Transposition

```
1.  procedure ODD-EVEN_PAR(n)
2.  begin
3.      id := process's label
4.      for i := 1 to n do
5.      begin
6.          if i is odd then
7.              if id is odd then
8.                  compare-exchange_min(id + 1);
9.              else
10.                 compare-exchange_max(id − 1);
11.         if i is even then
12.             if id is even then
13.                 compare-exchange_min(id + 1);
14.             else
15.                 compare-exchange_max(id − 1);
16.     end for
17. end ODD-EVEN_PAR
```

Parallel formulation of odd-even transposition.

Parallel Odd-Even Transposition

- Consider a block of n/p elements per processor.
- The first step is a local sort.
- In each subsequent step, the compare exchange operation is replaced by the compare split operation.
- The parallel run time of the formulation is

$$T_P = \overbrace{\Theta\left(\frac{n}{p}\log\frac{n}{p}\right)}^{\text{local sort}} + \overbrace{\Theta(n)}^{\text{comparisons}} + \overbrace{\Theta(n).}^{\text{communication}}$$

Parallel Odd-Even Transposition

- The parallel formulation is cost-optimal for $p = O(\log n)$.
- The isoefficiency function of this parallel formulation is $\Theta(p2^p)$.

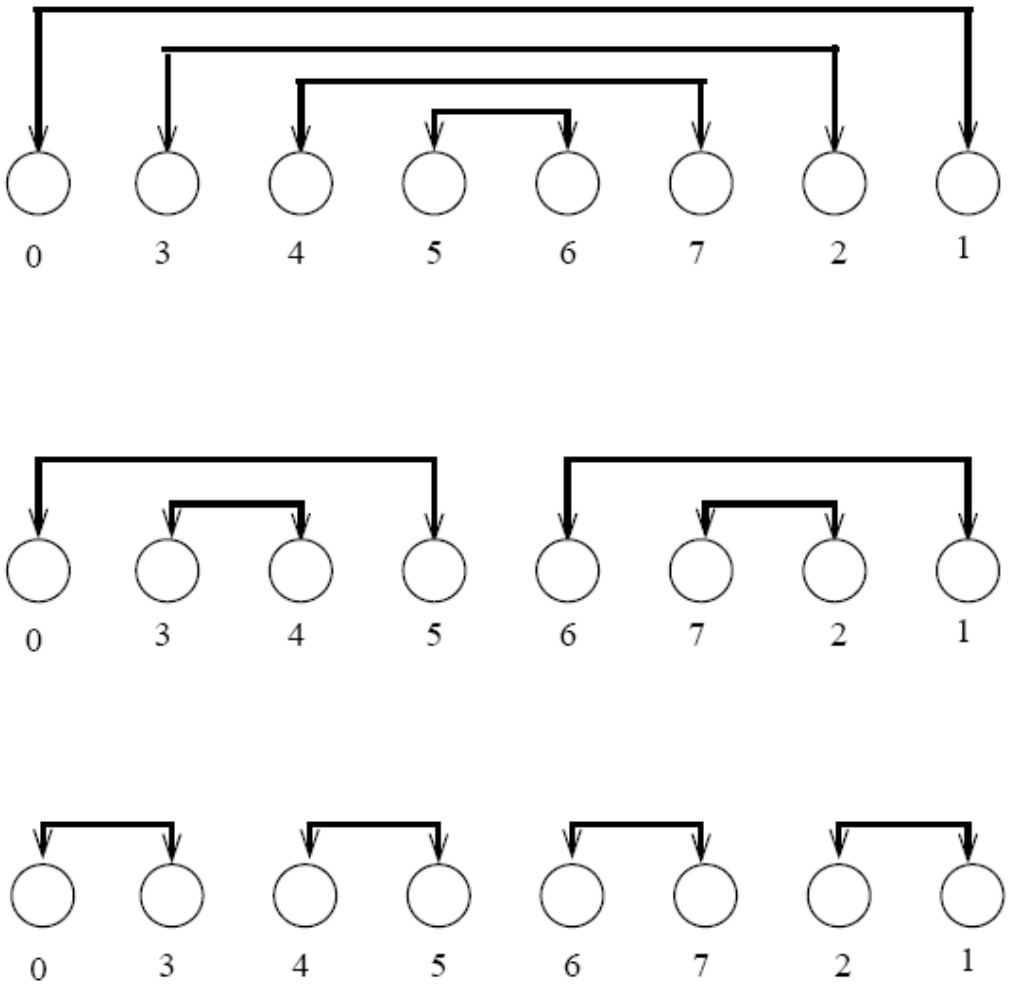
Shellsort

- Let n be the number of elements to be sorted and p be the number of processes.
- During the first phase, processes that are far away from each other in the array compare-split their elements.
- During the second phase, the algorithm switches to an odd-even transposition sort.

Parallel Shellsort

- Initially, each process sorts its block of n/p elements internally.
- Each process is now paired with its corresponding process in the reverse order of the array. That is, process P_i, where $i < p/2$, is paired with process P_{p-i-1}.
- A compare-split operation is performed.
- The processes are split into two groups of size $p/2$ each and the process repeated in each group.

Parallel Shellsort



An example of the first phase of parallel shellsort on an eight-process array.

Parallel Shellsort

- Each process performs $d = \log p$ compare-split operations.
- With $O(p)$ bisection width, each communication can be performed in time $\Theta(n/p)$ for a total time of $\Theta((n \log p)/p)$.
- In the second phase, l odd and even phases are performed, each requiring time $\Theta(n/p)$.
- The parallel run time of the algorithm is:

$$T_P = \overbrace{\Theta\left(\frac{n}{p}\log\frac{n}{p}\right)}^{\text{local sort}} + \overbrace{\Theta\left(\frac{n}{p}\log p\right)}^{\text{first phase}} + \overbrace{\Theta\left(l\frac{n}{p}\right)}^{\text{second phase}}. \qquad (3)$$

Assignment

Q.1)Discuss bubble sot & its variant.

Q.2)Explain parallel shell sort/

Quicksort

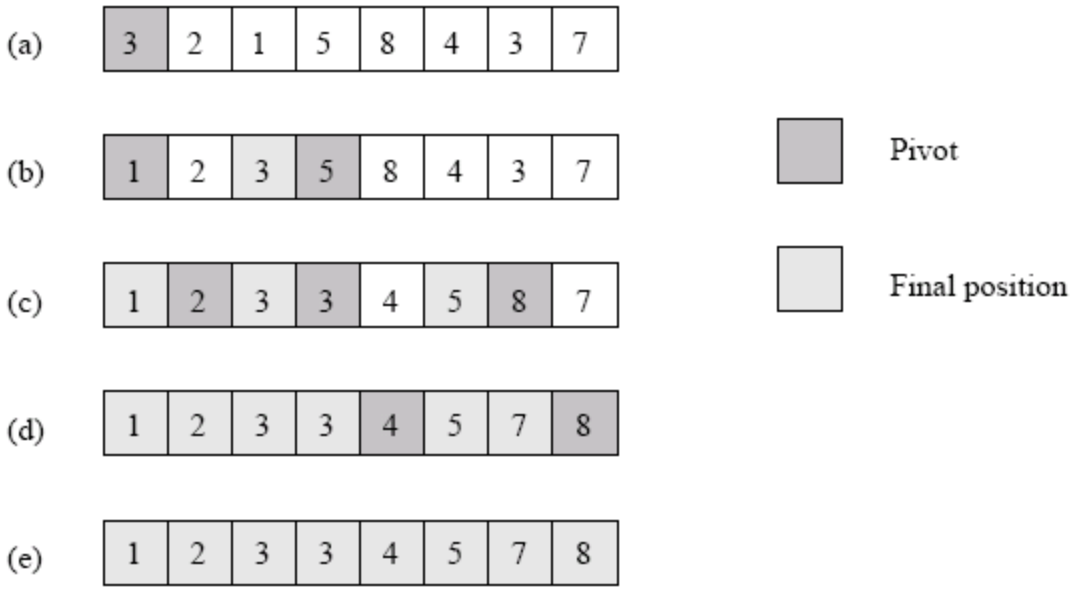
- Quicksort is one of the most common sorting algorithms for sequential computers because of its simplicity, low overhead, and optimal average complexity.
- Quicksort selects one of the entries in the sequence to be the pivot and divides the sequence into two - one with all elements less than the pivot and other greater.
- The process is recursively applied to each of the sublists.

Quicksort

```
1.   procedure QUICKSORT (A, q, r)
2.   begin
3.       if q < r then
4.       begin
5.           x := A[q];
6.           s := q;
7.           for i := q + 1 to r do
8.               if A[i] ≤ x then
9.               begin
10.                  s := s + 1;
11.                  swap(A[s], A[i]);
12.              end if
13.          swap(A[q], A[s]);
14.          QUICKSORT (A, q, s);
15.          QUICKSORT (A, s + 1, r);
16.      end if
17.  end QUICKSORT
```

The sequential quicksort algorithm.

Quicksort



Example of the quicksort algorithm sorting a sequence of size $n = 8$.

Quicksort

- The performance of quicksort depends critically on the quality of the pivot.
- In the best case, the pivot divides the list in such a way that the larger of the two lists does not have more than an elements (for some constant a).
- In this case, the complexity of quicksort is $O(n \log n)$.

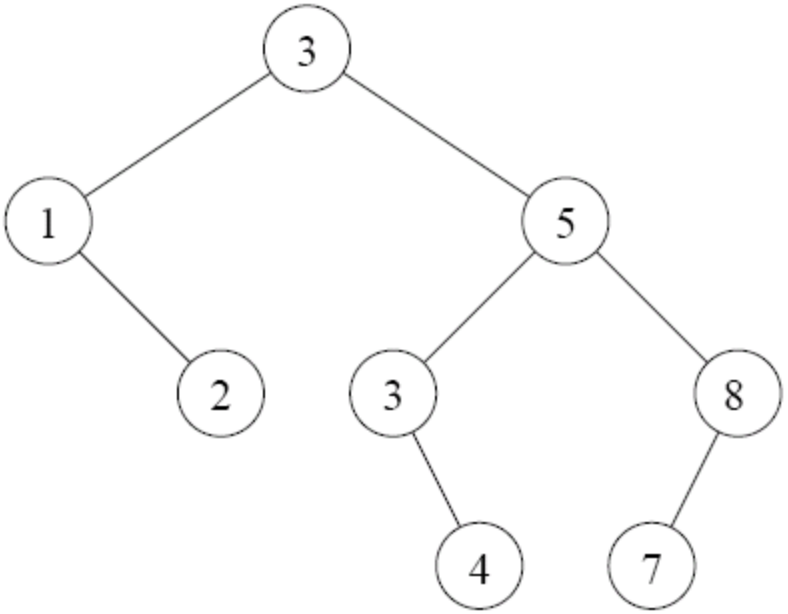
Parallelizing Quicksort

- Lets start with recursive decomposition - the list is partitioned serially and each of the subproblems is handled by a different processor.
- The time for this algorithm is lower-bounded by $\Omega(n)$!
- Can we parallelize the partitioning step - in particular, if we can use n processors to partition a list of length n around a pivot in $O(1)$ time, we have a winner.
- This is difficult to do on real machines, though.

Parallelizing Quicksort: PRAM Formulation

- We assume a CRCW (concurrent read, concurrent write) PRAM with concurrent writes resulting in an arbitrary write succeeding.
- The formulation works by creating pools of processors. Every processor is assigned to the same pool initially and has one element.
- Each processor attempts to write its element to a common location (for the pool).
- Each processor tries to read back the location. If the value read back is greater than the processor's value, it assigns itself to the `left' pool, else, it assigns itself to the `right' pool.
- Each pool performs this operation recursively.
- Note that the algorithm generates a tree of pivots. The depth of the tree is the expected parallel runtime. The average value is $O(\log\ n)$.

Parallelizing Quicksort: PRAM Formulation



A binary tree generated by the execution of the quicksort algorithm. Each level of the tree represents a different array-partitioning iteration. If pivot selection is optimal, then the height of the tree is $\Theta(\log\ n)$, which is also the number of iterations.

Parallelizing Quicksort: PRAM Formulation

(a)

1	2	3	4	5	6	7	8
33	21	13	54	82	33	40	72

(b) root = 4

(c)

	1	2	3	4	5	6	7	8
leftchild				1				
rightchild				5				

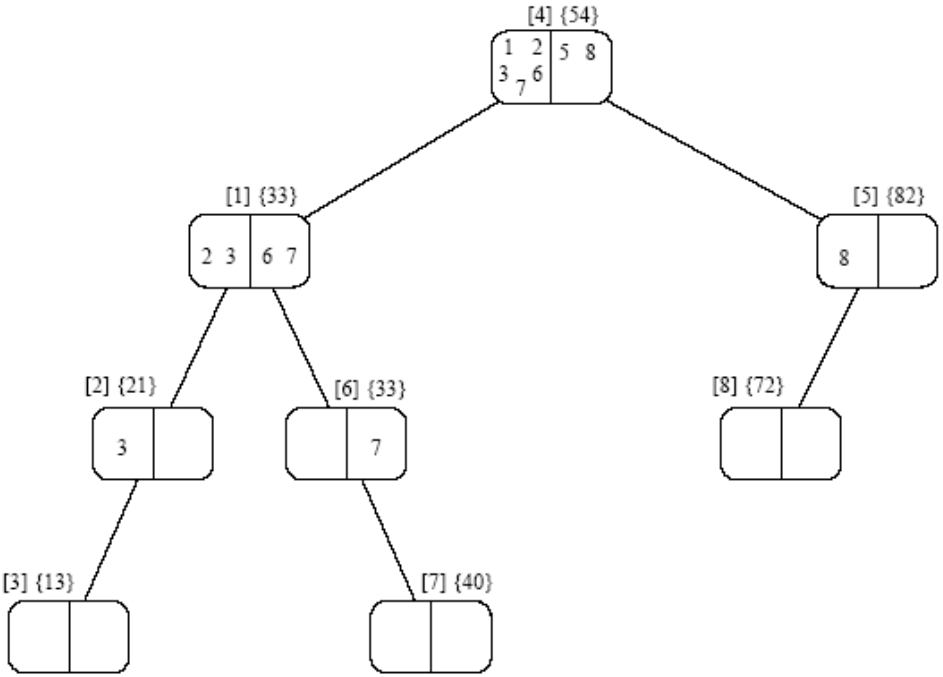
(d)

	1	2	3	4	5	6	7	8
leftchild	2			1	8			
rightchild	6			5				

(e)

	1	2	3	4	5	6	7	8
leftchild	2	3		1	8			
rightchild	6			5		7		

(f)

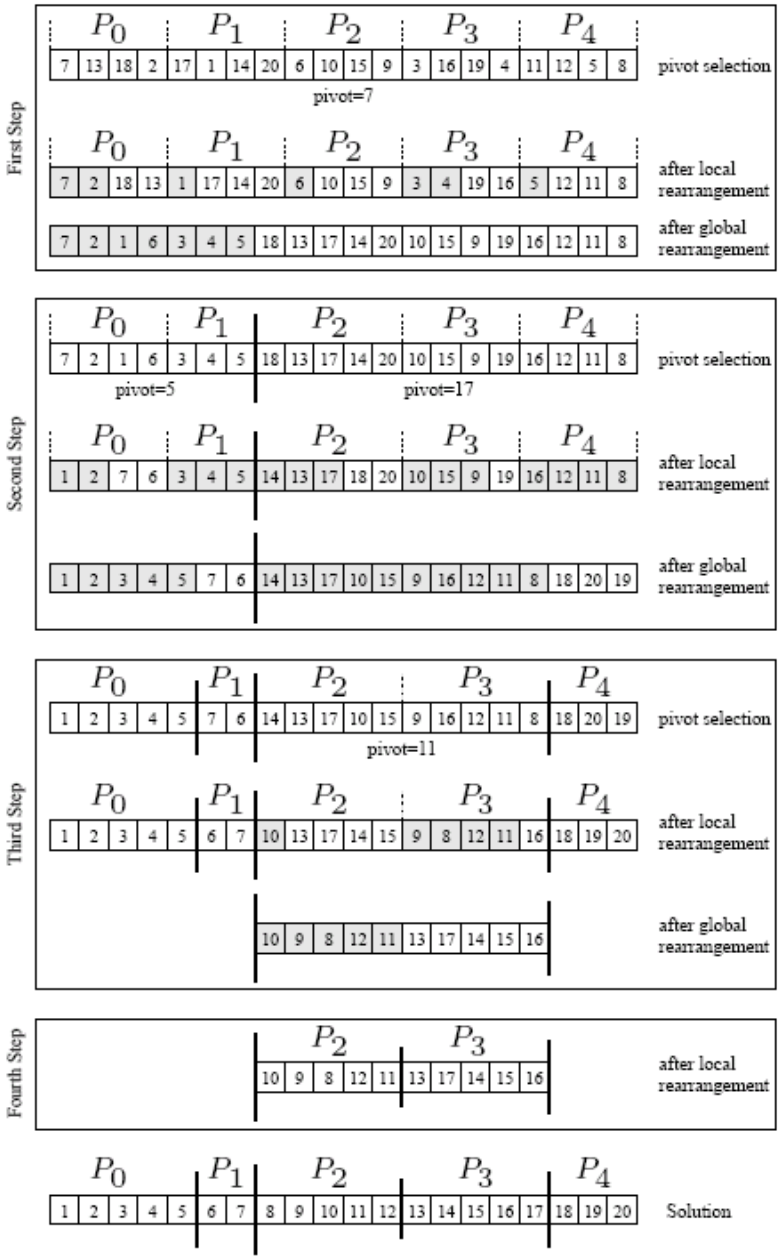


The execution of the PRAM algorithm on the array shown in (a).

Parallelizing Quicksort: Shared Address Space Formulation

- Consider a list of size n equally divided across p processors.
- A pivot is selected by one of the processors and made known to all processors.
- Each processor partitions its list into two, say L_i and U_i, based on the selected pivot.
- All of the L_i lists are merged and all of the U_i lists are merged separately.
- The set of processors is partitioned into two (in proportion of the size of lists L and U). The process is recursively applied to each of the lists.

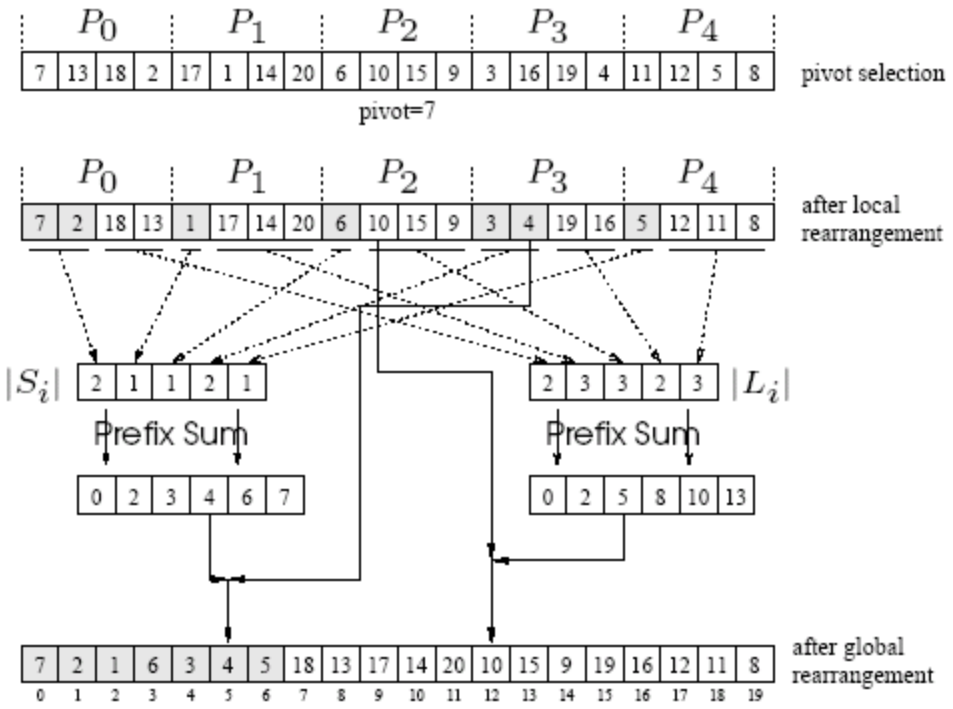
Shared Address Space Formulation



Parallelizing Quicksort: Shared Address Space Formulation

- The only thing we have not described is the global reorganization (merging) of local lists to form L and U.
- The problem is one of determining the right location for each element in the merged list.
- Each processor computes the number of elements locally less than and greater than pivot.
- It computes two sum-scans to determine the starting location for its elements in the merged L and U lists.
- Once it knows the starting locations, it can write its elements safely.

Parallelizing Quicksort: Shared Address Space Formulation



Efficient global rearrangement of the array.

Parallelizing Quicksort: Shared Address Space Formulation

- The parallel time depends on the split and merge time, and the quality of the pivot.
- The latter is an issue independent of parallelism, so we focus on the first aspect, assuming ideal pivot selection.
- The algorithm executes in four steps: (i) determine and broadcast the pivot; (ii) locally rearrange the array assigned to each process; (iii) determine the locations in the globally rearranged array that the local elements will go to; and (iv) perform the global rearrangement.
- The first step takes time $\Theta(\log p)$, the second, $\Theta(n/p)$, the third, $\Theta(\log p)$, and the fourth, $\Theta(n/p)$.
- The overall complexity of splitting an n-element array is $\Theta(n/p) + \Theta(\log p)$.

Parallelizing Quicksort: Shared Address Space Formulation

- The process recurses until there are p lists, at which point, the lists are sorted locally.
- Therefore, the total parallel time is:

$$T_P = \overbrace{\Theta\left(\frac{n}{p}\log\frac{n}{p}\right)}^{\text{local sort}} + \overbrace{\Theta\left(\frac{n}{p}\log p\right) + \Theta(\log^2 p)}^{\text{array splits}}. \qquad (4)$$

- The corresponding isoefficiency is $\Theta(p\log^2 p)$ due to broadcast and scan operations.

Parallelizing Quicksort: Message Passing Formulation

- A simple message passing formulation is based on the recursive halving of the machine.
- Assume that each processor in the lower half of a p processor ensemble is paired with a corresponding processor in the upper half.
- A designated processor selects and broadcasts the pivot.
- Each processor splits its local list into two lists, one less (L_i), and other greater (U_i) than the pivot.
- A processor in the low half of the machine sends its list U_i to the paired processor in the other half. The paired processor sends its list L_i.
- It is easy to see that after this step, all elements less than the pivot are in the low half of the machine and all elements greater than the pivot are in the high half.

Parallelizing Quicksort: Message Passing Formulation

- The above process is recursed until each processor has its own local list, which is sorted locally.
- The time for a single reorganization is $\Theta(\log\ p)$ for broadcasting the pivot element, $\Theta(n/p)$ for splitting the locally assigned portion of the array, $\Theta(n/p)$ for exchange and local reorganization.
- We note that this time is identical to that of the corresponding shared address space formulation.
- It is important to remember that the reorganization of elements is a bandwidth sensitive operation.

Bucket and Sample Sort

- In Bucket sort, the range $[a,b]$ of input numbers is divided into m equal sized intervals, called buckets.
- Each element is placed in its appropriate bucket.
- If the numbers are uniformly divided in the range, the buckets can be expected to have roughly identical number of elements.
- Elements in the buckets are locally sorted.
- The run time of this algorithm is $\Theta(n\log(n/m))$.

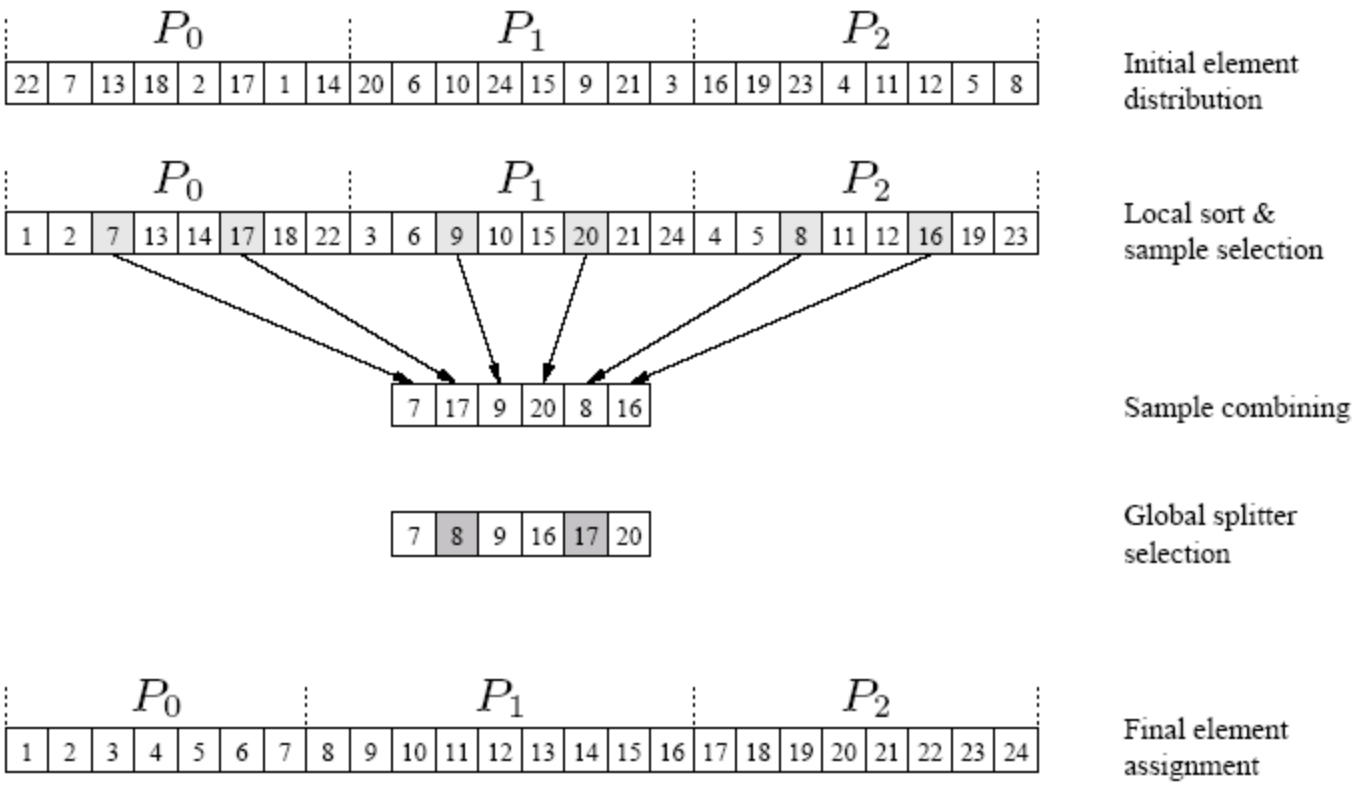
Parallel Bucket Sort

- Parallelizing bucket sort is relatively simple. We can select $m = p$.
- In this case, each processor has a range of values it is responsible for.
- Each processor runs through its local list and assigns each of its elements to the appropriate processor.
- The elements are sent to the destination processors using a single all-to-all personalized communication.
- Each processor sorts all the elements it receives.

Parallel Bucket and Sample Sort

- The critical aspect of the above algorithm is one of assigning ranges to processors. This is done by suitable splitter selection.
- The splitter selection method divides the n elements into m blocks of size n/m each, and sorts each block by using quicksort.
- From each sorted block it chooses $m-1$ evenly spaced elements.
- The $m(m-1)$ elements selected from all the blocks represent the sample used to determine the buckets.
- This scheme guarantees that the number of elements ending up in each bucket is less than $2n/m$.

Parallel Bucket and Sample Sort



An example of the execution of sample sort on an array with 24 elements on three processes.

Parallel Bucket and Sample Sort

- The splitter selection scheme can itself be parallelized.
- Each processor generates the $p-1$ local splitters in parallel.
- All processors share their splitters using a single all-to-all broadcast operation.
- Each processor sorts the $p(p-1)$ elements it receives and selects $p-1$ uniformly spaces splitters from them.

Parallel Bucket and Sample Sort: Analysis

- The internal sort of n/p elements requires time $\Theta((n/p)\log(n/p))$, and the selection of $p-1$ sample elements requires time $\Theta(p)$.
- The time for an all-to-all broadcast is $\Theta(p^2)$, the time to internally sort the $p(p-1)$ sample elements is $\Theta(p^2 \log\ p)$, and selecting $p-1$ evenly spaced splitters takes time $\Theta(p)$.
- Each process can *insert* these $p-1$splitters in its local sorted block of size n/p by performing $p-1$ binary searches in time $\Theta(p\log(n/p))$.
- The time for reorganization of the elements is $O(n/p)$.

Parallel Bucket and Sample Sort: Analysis

- The total time is given by:

$$T_P = \overbrace{\Theta\left(\frac{n}{p}\log\frac{n}{p}\right)}^{\text{local sort}} + \overbrace{\Theta\left(p^2\log p\right)}^{\text{sort sample}} + \overbrace{\Theta\left(p\log\frac{n}{p}\right)}^{\text{block partition}} + \overbrace{\Theta(n/p).}^{\text{communication}} \quad (5)$$

- The isoefficiency of the formulation is $\Theta(p^3 \log\ p)$.